Relations

For example,

- a, b real numbers, then we write

  $a < b$ "a is strictly less than b"

  $<$ denotes a relation between two numbers.

- a, b integers.

  $a \mid b$ "~~b is a~~ There is some $c \in \mathbb{Z}$, $b = ac$"

- a real number, S a set

  $a \in S$ "a is an element of S"

- a, b integers. n a natural number

  $a \equiv b \pmod{n}$

Each encodes a relationship between two (possibly different kinds of) objects

Example: Consider the set $A = \{2, 4, 6\}$

If our relationship is "Is $a < b$?"

$$
\begin{array}{ccc}
2 \not< 2 & 2 < 4 & 2 < 6 \\
4 \not< 2 & 4 \not< 4 & 4 < 6 \\
6 \not< 2 & 6 \not< 4 & 6 \not< 6
\end{array}
$$

The relationship is fully described by knowing which pairs $(a,b) \in A \times A$ satify $a < b$, and which don't.

Def: Let A and B be any two sets.
A relation from A to B is a subset $R \subseteq A \times B$.

We write aRb if (a,b) is an element of R

Example: Up above, $A = \{2,4,6\}$ $B = \{2,4,6\}$ $R = \{(2,4), (2,6), (4,6)\}$

<u>Comment</u>: In most cases, we will have $A = B$

In this situation, we call R a relation on the set A.

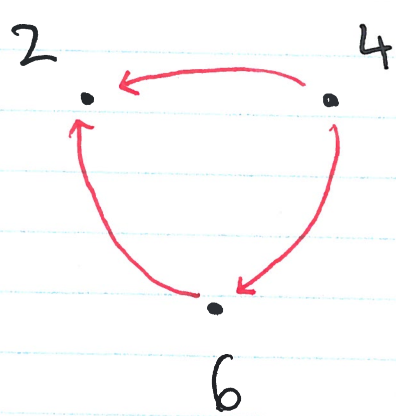
<u>Ex:</u> $A = B = \{2, 4, 6\}$. For $a, b \in A$

Relation: aRb "does the number a come earlier in the English dictionary?"

two	$2 \not R 2$	$4 R 2$	$6 R 2$
four	$2 \not R 4$	$4 \not R 4$	$6 \not R 4$
six	$2 \not R 6$	$4 R 6$	$6 \not R 6$

so $R = \{(4,2), (6,2), (4,6)\}$.

Another way to encode this:

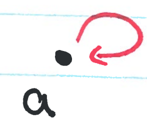


Equivalence Relations:

Let R is a relation from a set A to itself.

Def: We say R is reflexive

if $\forall a \in A,\ aRa$



For example: the relation $a|b$ for integers is reflexive, because $a|a$.

The relation $a>b$ is not! because $a \not> a$.

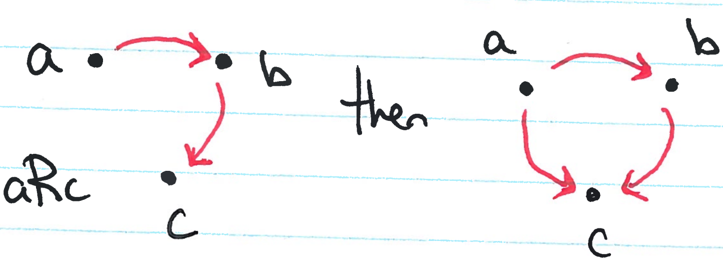
Def: We say R is symmetric

if $\forall a, b, \in A$

if aRb then bRa

Def: We say R is transitive

if $\forall a, b, c \in A$



if aRb and bRc, then aRc

Example: Consider $A = \mathbb{Z}$, and the relation

$$R = \{(a,b) : a \mid b\}.$$

Q: Is this reflexive? Symmetric? Transitive?

✓ ✗ ✓

- For every integer a, $a \mid a$.
- There exist $a, b \in \mathbb{Z}$ s.t. $a \mid b$ and $b \nmid a$. For example $a = 2, b = 6$.
- It is transitive, because if $a \mid b$ and $b \mid c$, then $a \mid c$.

This has been a useful property for us!

Def: A relation R on the set A is called an equivalence relation if it's reflexive, symmetric, and transitive.

Example: Let $A = \mathbb{Z}$ and pick any $n \in \mathbb{N}$. Then the relation $a \equiv b \pmod{n}$ is an equivalence relation.

Exercise: For each case below, reflexive?
symmetric?
transitive?

- $A = \{\text{students at UBC}\}$

reflexive ✓
symmetric ✓
transitive ✗

aRb if a and b attended ~~went to~~ high school together at some point

- $A = \{\text{people in this room}\}$

reflexive ✓
symmetric ✓
transitive ✗

aRb if a and b are within 1 meter of each other

- $A = \mathbb{Z}$

aRb if $\gcd(a,b) > 1$.

• not reflexive! Because $\gcd(1,1)$ is not greater than 1

• symmetric ✓

• not transitive! Because, for example, $\gcd(2,6) = 2 > 1$ $\gcd(6,3) = 3 > 1$ but $\gcd(2,3) = 1$.

- $A = \mathbb{R}$

aRb if $a - b$ is an integer.

- reflexive ✓
- symmetric ✓
- transitive ✓